Phase space Feynman path integrals of parabolic type

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In this talk, we introduce our results for the phase space path integrals of higher-order parabolic type on $\mathbb{R}^d$ ([2][3][4]) and on the torus $\mathbb{T}^d$ with $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ ([1]).

Let $0<T\leq\mathcal{T}<\infty$ and $x\in\mathbb{R}^d$. Let $U(T,0)$ be the fundamental solution for the higher-order parabolic equation

$$\Big(\partial_T+H(T,x,-i\partial_x)\Big)U(T,0)v(x)=0\,,\quad U(0,0)v(x)=v(x)\,. \tag{1}$$

By the Fourier transform w.r.t. $x_0\in\mathbb{R}^d$ and the inverse Fourier transform w.r.t. $\xi_0\in\mathbb{R}^d$, the pseudo-differential operator $H(T,x,-i\partial_x)$ on $\mathbb{R}^d$ can be written as

$$H(T,x,-i\partial_x)v(x)=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}}e^{i(x-x_0)\cdot\xi_0}H(T,x,\xi_0)v(x_0)dx_0d\xi_0 \tag{2}$$

with the function $H(T,x,\xi_0)$. We consider the function $U(T,0,x,\xi_0)$ such that

$$U(T,0)v(x)\equiv\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}}e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0\,. \tag{3}$$

As an approximation of $U(T,0)v(x)$, we use

$$I(T,0)v(x)\equiv\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}}e^{i(x-x_0)\cdot\xi_0}e^{-\int_0^T H(t,x,\xi_0)dt}v(x_0)dx_0d\xi_0\,. \tag{4}$$

For any division $\Delta_{T,0}:T=T_{J+1}>T_J>\cdots>T_1>T_0=0$, we have

$$U(T,0)v(x)=U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)v(x)\,. \tag{5}$$

Let $t_j=T_j-T_{j-1}$ and $|\Delta_{T,0}|=\max_{1\leq j\leq J+1}t_j$. Set $x_{J+1}=x$. Let $x_j,\ \xi_j\in\mathbb{R}^d$. In [4], under a suitable condition, using $I(T_j,T_{j-1})$ as an approximation of $U(T_j,T_{j-1})$ as $|\Delta_{T,0}|\to 0$, we have

$$\begin{aligned}U(T,0)v(x)&=\lim_{|\Delta_{T,0}|\to0}I(T,T_J)I(T_J,T_{J-1})\cdots I(T_2,T_1)I(T_1,0)v(x)\\&=\lim_{|\Delta_{T,0}|\to0}\left(\frac{1}{2\pi}\right)^{d(J+1)}\int_{\mathbb{R}^{2d(J+1)}}e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j}H(t,x_j,\xi_{j-1})dt)}v(x_0)\prod_{j=0}^{J}dx_jd\xi_j.\end{aligned} \tag{6}$$

Furthermore, by (3) and (6), we get

$$\begin{aligned}&e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)\\&=\lim_{|\Delta_{T,0}|\to0}\left(\frac{1}{2\pi}\right)^{dJ}\int_{\mathbb{R}^{2dJ}}e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j}H(t,x_j,\xi_{j-1})dt)}\prod_{j=1}^{J}dx_jd\xi_j\,.\end{aligned} \tag{7}$$

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According Feynman's idea, we introduce the position path $q(t)$ with $q(T_j) = x_j$ and the momentum path $p(t)$ with $p(T_j) = \xi_j$. Then we can formally write (7) as

$$e^{i(x-x_0)\cdot\xi_0} U(T,0,x,\xi_0) = \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} \mathcal{D}(q,p)\,, \tag{8}$$

where

$$\phi(q,p) = \int_{[0,T)} p(t)\cdot dq(t) + i\int_{[0,T)} H(t,q(t),p(t))dt \tag{9}$$

is the action and the path integral is a sum over all the paths (q,p). The expression (7) is called the time slicing approximation of the path integral (8).

In [2][3], using the time slicing approximation via piecewise constant paths, we give a mathematically rigorous meaning to the phase space path integrals

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} F(q,p)\mathcal{D}(q,p) \tag{10}$$

of higher-order parabolic type with the general functional $F(q,p)$ as integrand. We can regard (8) as the case of (10) with $F(q,p)\equiv 1$. More precisely, we give two general sets $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals such that for any $F(q,p)\in\mathcal{F}_{\mathcal{Q}}\cup\mathcal{F}_{\mathcal{P}}$, the time slicing approximation of (10) converges uniformly on compact subsets with respect to the final point x of position paths and the initial point ξ_0 of momentum paths. Each of the two sets $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ is closed under addition, multiplication, translation, invertible linear transformation and functional differentiation. Therefore, we can produce many functionals $F(q,p)$ which are path integrable. Furthermore, though we must pay attention for use, we give the properties of the path integrals (10) similar to some properties of the standard integrals.

On the other hand, by the Fourier series expansion w.r.t. $x_0\in\mathbb{T}^d$ and $\xi_0\in\mathbb{Z}^d$, the pseudo-differential operator $H(T,x,-i\partial_x)$ on the torus $\mathbb{T}^d$ can be written as

$$H(T,x,-i\partial_x)v(x) = \left(\frac{1}{2\pi}\right)^d \sum_{\xi_0\in\mathbb{Z}^d}\int_{\mathbb{T}^d} e^{i(x-x_0)\cdot\xi_0} H(T,x,\xi_0)v(x_0)dx_0\,. \tag{11}$$

In [1], using (11) instead of (2), we treat the path integral on the torus $\mathbb{T}^d$.

References

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